

Superconductive proximity in a Topological Insulator slab and excitations bound to an axial vortex

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We consider the proximity effect in a Topological Insulator sandwiched between two conventional superconductors, by comparing *s* – wave spin singlet superconducting pairing correlations and *odd* – parity triplet pairing correlations with zero spin component orthogonal to the slab (“polar” phase). A superconducting gap opens in the Dirac dispersion of the surface states existing at the interfaces. An axial vortex is included, piercing the slab along the normal to the interfaces with the superconductors. It is known that, when proximity is *s* – wave, quasiparticles in the gap are Majorana Bound States, localized at opposite interfaces. We report the full expression for the quantum field associated to the midgap neutral fermions, as derived in the two-orbital band model for the TI. When proximity involves *odd* – parity pairing, midgap modes are charged Surface Andreev Bound States, and they originate from interfacial circular states of definite chirality, centered at the vortex singularity and decaying in the TI film with oscillations. When the chemical potential is moved away from midgap, extended states along the vortex axis are also allowed. Their orbital structure depends on the symmetry of the bulk band from where the quasiparticle level splits off.

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I. INTRODUCTION

Topological insulators (TIs), bulk insulators, with metallic surface states protected by time reversal invariance, are attracting widespread interest for potential applications in nanoelectronics and spintronics^{1,2}. Three-dimensional (3D) TIs, such as Bi_2Se_3 or Bi_2Te_3 ^{3,4}, have two dimensional (2D) metallic surface states, whose band structure consists of an odd number of Dirac cones, centered at Time Reversal (TR) invariant momenta in the surface Brillouin Zone (BZ). Surface sensitive experiments such as angle-resolved photoemission spectroscopy^{5,6} and scanning tunneling microscopy (STM)⁷ have confirmed the existence of these exotic surface metal states with a Dirac-like energy dispersion.

Boundary states can be characterized with the help of minimal models for the various geometries, whose corresponding Hamiltonians are classified generalizing Altland-Zirnbauer symmetry classes⁸ and the topological invariants in mapping from the configurational space, parametrized by the k – vectors, to the real space. Surface Dirac fermions can become chiral when time reversal is broken by (e.g.) ferromagnetic covering and they can be bound to defects such as magnetic domain walls. The zoology of all the possible boundary states available has been presented in full in Ref.s 9,10. A model which successfully describes e.g. the (1,1,1) surface of the (otherwise centrosymmetric) Bi_2Se_3 close to the Γ point of the surface BZ has been introduced in Ref. 4 and can be taken as the prototype model for TR invariant TIs. It is based on the continuum limit for long wavelengths and includes appropriately the spin-orbit interaction which is the crucial feature for the topological protection. It is a two band model involving two species of orbitals, of even

and odd parity, with a bulk gap $2M$.

We consider a TI film terminated at an $x - y$ plane, sandwiched between two superconductors. Here we choose \hat{z} perpendicular to the quintuple layers¹¹. The metallic states at the interfaces become superconducting by proximity effect. The nature of the induced superconductivity when the chemical potential is within the bulk gap is unknown. Solving this puzzle is extremely important in view of the possible existence of Majorana Bound States (MBSs) localized at interfaces between conventional superconductors and TIs¹². It has been argued that induced singlet, *s* – wave superconducting correlations, with order parameter Δ_s , could turn the pairing of the boundary states (with $M > \mu \gg \Delta_s$) into an effective spinless *odd* – parity pairing of the *p* – wave type: $p_x - ip_y$ ¹³. In this case a vortex piercing the structure can bind a Majorana fermion at each interface^{13–16}. It has been shown that, for the case of an *s* – wave pairing, in presence of an axial vortex, the zero energy Majorana Fermion can survive even when μ is moved within one of the bulk bands by p- or n-doping, till the spin-orbit coupling is able to supply a Berry phase of π to its wavefunction¹⁷. Roughly speaking, the minus sign that the Majorana fermion acquires in turning once around a loop, which is intrinsic to its fermionic character, should be compensated by the spin-orbit texture, in order to provide a single valued solution of the eigenvalue problem.

Still, there is no evidence that superconductivity induced in the boundary states would be of the singlet, *s* – wave type.

It has been found that few per cent *Cu* doping of Bi_2Se_3 bulk crystals moves the chemical potential μ within the bulk conduction band and the superconduct-

ing phase transition takes place at $T_c \sim 3.8K^{7,18}$. Odd-parity pairing seems to be favored by strong spin-orbit coupling¹⁹. Topological odd-parity superconductors are fully gapped in the bulk but have gapless Surface Andreev Bound States (SABS). It has been noticed recently that a mirror symmetry (composed of the odd mirror reflection in the yz plane $x \rightarrow -x$ and of a Z_2 gauge transformation $\Delta \rightarrow -\Delta$) characterizes the helicity of the SABS^{2,20}. Superconductivity induced in the boundary states of a TI slab by a singlet superconductor, odd with respect to the mirror plane $z = 0$ could be of the "polar" type, as explained below, with an order parameter proportional to p_z and a vanishing spin projection onto the z spin quantization axis, which is pinned to the normal to the slab by the spin-orbit interaction. Strictly speaking, such a superconducting order would have a nodal line of excitations exactly at the $x - y$ plane, which allows for the sign inversion at the boundary. However state of the art TI flakes are known to be plagued by impurities and vacancies in the bulk and it is likely that bulk impurity bands arise with little dispersion which provide a non vanishing Fermi momentum p_F also in the z -direction. Therefore, we argue that the proximity gap never closes, even when μ is located within the bulk gap. Otherwise, the presence of the gap nodal line at the surface would affect the lifetime of the states bound at a vortex core, but not their topological origin.

In this paper we explore the nature of the zero energy excitations corresponding to quasiparticles bound to an axial vortex piercing a sandwich $S/TI/S$ geometry and carrying a magnetic flux. While topological superconductors can be well classified when TR invariance is preserved^{21,22}, by the breaking of TR symmetry the topological protection can be washed out. We compare the two kind of superconducting induced orders in the limit $M > \mu \gg \Delta$: a) an s -wave induced order parameter, b) a p_z -wave (polar), odd parity order parameter.

The two-orbital model appears to be adequate for both the cases. Using this model we find that a proximity superconducting gap opens at μ located in the Dirac cone dispersion of the boundary states. A vortex piercing the slab binds zero energy excitations inside the superconducting gap depending on the value of μ . However, their nature is strongly dependent on the type of superconducting ordering, a) or b), induced by proximity:

a) Majorana zero energy excitations are bound to the vortex as expected¹³ (called MBS henceforth). A neutral quasiparticle is localized at each of the interfaces with the superconductors and we present an analytical explicit expression for the quantum field that describes the excitation in the two-orbital model in the simple case of $\mu = 0$ (midgap state). Orbital parities are mixed and spin is not conserved. The two spacially separated modes are TR mates, no matter that the vortex breaks TR symmetry. It has been argued that even moving the chemical potential inside the bulk band the vortex could host a MBS¹⁷.

b) In the case of odd parity topological superconductivity, with μ inside the bulk bands, a linear dispersion of SABS arises in the superconducting gap, close to $k = 0$ ²⁰. These states are MBS. In this work we show that when μ is in the bulk gap, a vortex hosts SABS, as well. However they are no longer neutral fermion excitations. By breaking the TR symmetry of the odd-parity topological superconductor, the vortex makes the zero energy SABS excitations turn into a pair of Dirac states, loosely bound to the vortex core. To prove our statement, we solve explicitly the $\mu = 0$ (midgap) case, in full analogy with the s -wave pairing case described in a). The modes, having odd and even orbital symmetry, respectively, propagate along the vortex axis with opposite chirality. At $\mu \neq 0$, if the chemical potential matches an energy split off from the bulk conduction or valence band, zero energy excitations exist, with mixed type of orbitals. Expectation value of the spin of these excitations is, in any case, zero.

The important novelty in our calculation is the assumption of polar order parameter for μ within the bulk gap in the two-orbital model. Polar order, i.e. triplet pairing with zero projection of the Cooper pair spin along the normal to the boundary plane, is expected to be preferred in view of strong tendency to in-plane helical transport induced by spin-orbit coupling. We fully account for the two bulk band structure in the TI and for the third dimension \hat{z} orthogonal to the plane of the slab. The orthogonal direction is crucial for the description of these bound states. The celebrated argument by Fu and Kane¹³ is based on an effective two-dimensional system describing the flat boundary. It is easy to see that its generalization is unable to account for the proximity in the polar state. This is what we shortly report on, in closing this Introduction.

The starting point of Fu and Kane¹³ is the Dirac dispersion of the surface electronic states, proximized by an s -wave superconductor. The model can be effectively reduced to spinless electrons with linear energy dispersion in a p -wave pairing field. Here we want to extend their argument to induced odd parity proximization.

The original argument can be rephrased as follows. We consider just an effective Dirac-like Hamiltonian for the surface states at the flat boundary at $z = 0$. The velocity v characterizes the linear dispersion. The proximized TI Hamiltonian, in the basis $[\psi_{k\uparrow}, \psi_{k\downarrow}, \psi_{-k\downarrow}^\dagger, -\psi_{-k\uparrow}^\dagger]$, is²³:

$$H_{TI} = \begin{pmatrix} -\mu & vk_- & & \hat{\Delta} \\ vk_+ & -\mu & & \\ & \hat{\Delta} & \mu & -vk_- \\ & & -vk_+ & \mu \end{pmatrix}, \quad (1)$$

with $k_{\pm} = k e^{\pm i \theta_k}$. In this basis, s -wave pairing takes the form $\hat{\Delta}_s = \begin{pmatrix} \Delta_s & 0 \\ 0 & \Delta_s \end{pmatrix}$ and the four energy eigenvalues are $\pm \sqrt{(\pm vk - \mu)^2 + \Delta_s^2}$. If we disregard the two

bands which are far off the chemical potential, it is easy to show that an effective BdG Hamiltonian can be written

in terms of spinless operators corresponding to spinorial wavefunctions:

$$c_{\vec{k}} \equiv \begin{bmatrix} \psi_{\vec{k}\uparrow} \\ \psi_{\vec{k}\downarrow} \end{bmatrix} \rightarrow \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ e^{i\theta_k} \end{bmatrix}; \quad c_{-\vec{k}}^\dagger \equiv \begin{bmatrix} \psi_{-\vec{k}\downarrow}^\dagger \\ -\psi_{-\vec{k}\uparrow}^\dagger \end{bmatrix} \rightarrow \frac{1}{\sqrt{2}} \begin{bmatrix} e^{-i\theta_{-k}} \\ -1 \end{bmatrix}. \quad (2)$$

The new effective Nambu Hamiltonian reads:

$$H^{new} = \frac{1}{2} \begin{pmatrix} c_{\vec{k}}^\dagger & -c_{-\vec{k}} \end{pmatrix} \begin{pmatrix} [vk - \mu] & \Delta_s e^{-i\theta_k} \\ \Delta_s e^{i\theta_k} & -[vk - \mu] \end{pmatrix} \begin{pmatrix} c_{\vec{k}} \\ -c_{-\vec{k}}^\dagger \end{pmatrix} \quad (3)$$

Indeed, e.g., the matrix element h_{12}^{new} , in the basis given above is:

$$h_{12}^{new} = \frac{1}{2} (1 \ e^{-i\theta_k}) \begin{pmatrix} \Delta_s & 0 \\ 0 & \Delta_s \end{pmatrix} \begin{pmatrix} e^{-i\theta_k} \\ 1 \end{pmatrix} = \Delta_s e^{-i\theta_k}.$$

Eq.(3) provides an effective Hamiltonian for spinless particles with $\Delta_{\vec{k}} \propto k_x - ik_y$, which implies an effective p -wave pairing. According to ref.14,15, a vortex can sustain a MBS in such a system.

On the other hand, the odd-parity mean field coupling $\sum_{\sigma\sigma'} \Delta_{\sigma'\sigma} \psi_{\vec{k}\sigma}^\dagger \psi_{-\vec{k}\sigma'}^\dagger + h.c.$ is described by a vector $\vec{d}(\vec{k})$ which is odd for $\vec{k} \rightarrow -\vec{k}$. The generic, uniform order parameter is a matrix in the spin space $\Delta_{\sigma\sigma'}(\vec{k}) = [\vec{d}(\vec{k}) \cdot (\vec{s} \text{ is } s_2)]_{\sigma\sigma'}$. In our basis, it reads: $\hat{\Delta}_p = \begin{pmatrix} d_z & d_x - id_y \\ d_x + id_y & -d_z \end{pmatrix}$. If $\vec{d}(\vec{k}) \propto \vec{k}$ itself, the effective Hamiltonian H^{new} is the same as in eq.(3), but with Δ_s replaced by $|d_{||}|$ with $d_{||} \equiv (d_x, d_y)$. In the model d_z disappears altogether. This result doesn't look convincing, because it seems to be an artifact due to the restriction of the model to the two-dimensional $x-y$ boundary plane. By contrast, in view of the strong spin-orbit locking between spin and momentum in this plane, it is likely that a polar "p-wave" phase is stabilized, with vanishing z -projection of the Cooper pair spin. This would imply an order parameter $\vec{d} \equiv (0, 0, d_z) \propto k_z \hat{z}$. In the bulk, such an order parameter would have a gap with a nodal line at the Fermi surface, exactly at $k_z = 0$, because $\text{Det}(\hat{\Delta}_p) \propto k_z^4$. However, this may not be the case close to the surface where k_z is undefined. Besides, state of art flakes of topological insulators (prepared, for instance, by Chemical Vapor Deposition) are not impurity free in the bulk, and we argue that a non vanishing Fermi wavevector in the z -direction is anyhow present in the bulk, thus providing a non vanishing superconducting gap also at the boundary surface. Given this choice for the order parameter, our heuristic approach leading to Eq.(3) would make, in any case, the superconducting order parameter disappear in H^{new} , what is suspicious.

This argument shows that, when odd parity symmetry for the superconducting order parameter is chosen, with

vanishing spin component of the pair in the direction orthogonal to the surface, a full approach which includes the third dimension is mandatory.

The structure of the paper is the following. In Sec. II we first introduce the two-orbital model in the long wavelength continuum limit, close to the Γ point, which is of the Bogolubov-De Gennes type²⁴. The model applies to a bulk crystal with superconducting pairing correlations. It will be convenient to deal with the *odd-parity* order parameter case first, and rephrase the model for the *s-wave* pairing. This is done in Sec. III, where we derive the opening of the superconducting gap at the chemical potential, which can be located within one of the two bulk bands (thus giving rise to what is called a topological superconductor, Eq.(11)), or within the bulk gap. In the latter case the superconducting gap opens in the energy dispersion of the boundary states which is Dirac cone like, Eq.(12). The neutral excitation bound to the vortex at the slab boundary in the case of *s-wave* proximity is derived analytically for in Appendix A and discussed in Sec. IV.A. In Sec. IV.B we report on the midgap ($\mu = 0$) charged excitations bound at the vortex core in the *odd-parity* pairing. Their field operator and its spacial dependance is derived in Appendix B. Extended states along the vortex core are also derived, for the case of $\mu \neq 0$. Results are summarized in Sec. V. From our analysis, the main conclusion that we draw is the observation that there are no MBS with *odd-parity* superconductive proximity and zero projection of the pair spin, in the two-orbital model.

II. MODEL HAMILTONIANS

We start from the Hamiltonian introduced by H. Zhang *et al.*⁴ to describe three dimensional layered systems like Bi_2Se_3 , Sb_2Te_3 , and Bi_2Te_3 . They use a 4-dimensional basis-space which is the direct product of the spin space and of the orbital parity space originating from p_z orbitals, denoted as: $\{|p_z^g, \uparrow\rangle, |p_z^u, \uparrow\rangle, |p_z^g, \downarrow\rangle, |p_z^u, \downarrow\rangle\}$. Here $g(u)$ denotes even (odd) parity for $\vec{k} \rightarrow -\vec{k}$. On such a basis, the model Hamiltonian is (we assume the Fermi

velocity v to be isotropic and we put it equal to unity here and in the following):

$$\hat{h}_o = -\mathcal{M}\sigma_3 + i\sigma_1(s_1\partial_x - s_2\partial_y) + i\sigma_3s_3\partial_z. \quad (4)$$

Here s_α and σ_a ($\alpha, a \in \{1, 2, 3\}$) are Pauli matrices in the spin and orbital space, respectively. The hat, over \hat{h}_o reminds of its 4x4 matrix structure. \mathcal{M} includes second order derivatives $\mathcal{M} = M(z) + C\partial_{\parallel}^2 + c\partial_z^2$, parallel to the flat $x-y$ boundary plane terminating the quintuple layer. M is half the bulk gap. Non trivial topology is guaranteed by the condition $M, C, c > 0$, which corresponds to the inversion of the bulk bands.

The Bogolubov-De Gennes (BdG) mean field Hamiltonian, in the presence of a superconducting pairing requires an 8 x 8 matrix structure whose compact form is:

$$H_{BdG}(\vec{k}) = \begin{pmatrix} \hat{h}_o & \hat{\Delta} \\ \hat{\Delta}^\dagger & -\hat{h}_o^* \end{pmatrix}, \quad (5)$$

where \hat{h}_o and $\hat{\Delta}$ are 4X4 matrices. In the BdG representation, even parity singlet pairing requires that $\hat{\Delta}_s^T(\vec{k}) = \hat{\Delta}_s(-\vec{k})$, while odd-parity triplet pairing satisfies $\hat{\Delta}_p^T(\vec{k}) = -\hat{\Delta}_p(-\vec{k})$ ²⁵.

In our case, given the Hamiltonian

$$H - \mu N = (h_0 - \mu)\tau_z + H_{pair} \quad (6)$$

where the τ_a Pauli matrices act in the Nambu space, the matrix structure of the off-diagonal pairing Hamiltonian H_{pair} can acquire different forms, depending on the order parameter symmetry. In the even parity, s-wave, singlet case, we define $\Delta_s = \langle \psi_{u\uparrow}\psi_{u\downarrow} \rangle = \langle \psi_{g\uparrow}\psi_{g\downarrow} \rangle = -\langle \psi_{u\downarrow}\psi_{u\uparrow} \rangle = -\langle \psi_{g\downarrow}\psi_{g\uparrow} \rangle$, assumed to be independent of the orbital. It gives rise to a pairing Hamiltonian

$$H_{pair}^s = -i(\Delta_s s_y \tau_+ + h.c.).$$

On the other hand, it has been proposed that, when doped with few percent Cu , the Bi_2Se_3 undergoes the superconducting phase transition with an odd-parity order parameter¹⁹. The pairing Hamiltonian for the polar ordering presented in the Introduction is

$$H_{pair}^p = (\Delta_p \sigma_y s_y \tau_+ + h.c.),$$

where $\Delta_p = \langle \psi_{u\uparrow}\psi_{g\downarrow} \rangle = -\langle \psi_{g\uparrow}\psi_{u\downarrow} \rangle = -\langle \psi_{g\downarrow}\psi_{u\uparrow} \rangle = \langle \psi_{u\downarrow}\psi_{g\uparrow} \rangle$ is the odd-parity orbital order parameter. The change in signs in the expectation values for Δ_p arises from the fact that the operators act on a triplet pair with zero spin projection along z .

III. TI BOUNDARY STATES

In this Section we discuss the opening of the superconducting gap induced by proximity in the TI, when the bulk gap M is much larger than the superconducting gap Δ . Depending on the doping, the chemical

potential μ can be within a bulk band with dispersion $\eta_{\vec{k}} \approx \pm\sqrt{M^2 + k^2}$, or in the gap of the insulator, where the electronic states localized at the boundaries have a Dirac like dispersion. Here we choose \hat{z} perpendicular to the quintuple layers¹¹, which would give helical boundary states in the surface plane. Spin orientation is expected to be in the plane, as well. The Hamiltonian for the two different pairing symmetries, which we have introduced in the previous Section and the corresponding energy spectrum is given here below.

Odd-parity pairing

Let us first write down the Hamiltonian for the odd-parity pairing. It is convenient for our purposes to choose the basis

$$B \equiv [\psi_{g\uparrow}, \psi_{u\downarrow}, \psi_{u\downarrow}^\dagger, -\psi_{g\uparrow}^\dagger, \psi_{u\uparrow}, \psi_{g\downarrow}, \psi_{g\downarrow}^\dagger, -\psi_{u\uparrow}^\dagger]^T. \quad (7)$$

A matrix acting on this basis has 4X4 blocks corresponding to components with exchanged parities (g, u) . We address these blocks with Pauli matrices denoted by T^a . Within each of these blocks, a Nambu particle-hole component structure arises, of the kind: $[(\psi_\uparrow, \psi_\downarrow), (\psi_\downarrow^\dagger, -\psi_\uparrow^\dagger)]^T$, addressed by Pauli matrices \mathcal{C}^α . The spin space is addressed by Pauli matrices S^σ ($a, \alpha, \sigma = 1, 2, 3$). In this representation, the particle-hole conjugation is defined by $\Xi = (\mathbf{1} \times \mathcal{C}^2 \times S^2) \mathcal{K}$, where \mathcal{K} is the complex conjugation of the wavefunctions. Explicitly, the application of Ξ onto the basis B gives, apart for \mathcal{K} : $\Xi B = [\psi_{g\uparrow}^\dagger, \psi_{u\downarrow}^\dagger, \psi_{u\downarrow}, -\psi_{g\uparrow}, \psi_{u\uparrow}^\dagger, \psi_{g\downarrow}^\dagger, \psi_{g\downarrow}, -\psi_{u\uparrow}]^T$. Time Reversal operator is $\Theta = [\mathbf{1} \times \mathbf{1} \times iS^2] \mathcal{K}$ ($\Theta^2 = -1$). This transforms the basis into $[\psi_{u\downarrow}, -\psi_{g\uparrow}, -\psi_{g\uparrow}^\dagger, -\psi_{u\downarrow}^\dagger, \psi_{g\downarrow}, -\psi_{u\uparrow}, -\psi_{u\uparrow}^\dagger, -\psi_{g\downarrow}^\dagger]^T$, apart for \mathcal{K} .

The full 8X8 Hamiltonian in the basis B looks like:

$$H^p = \begin{pmatrix} H_+^p & S^3 i\partial_z \\ S^3 i\partial_z & H_-^p \end{pmatrix}, \quad (8)$$

where

$$H_\pm^p = \begin{pmatrix} \mp\mathcal{M} - \mu & i(\partial_x - i\partial_y) & \pm\Delta_p & 0 \\ i(\partial_x + i\partial_y) & \pm\mathcal{M} - \mu & 0 & \pm\Delta_p \\ \pm\Delta_p^* & 0 & \pm\mathcal{M} + \mu & i(\partial_x - i\partial_y) \\ 0 & \pm\Delta_p^* & i(\partial_x + i\partial_y) & \mp\mathcal{M} + \mu \end{pmatrix}, \quad (9)$$

(the derivatives act on the ψ fields). An unitary transformation, by changing the basis, $\hat{B}^T \rightarrow [\psi_{g\uparrow}, \psi_{u\downarrow}, \psi_{g\downarrow}, \psi_{u\uparrow}, -\psi_{g\uparrow}^\dagger, \psi_{u\downarrow}^\dagger, \psi_{g\downarrow}^\dagger, -\psi_{u\uparrow}^\dagger]^T$, rewrites this Hamiltonian into the BdG form of Eq.(5). It can also be shown that another unitary transformation maps the Hamiltonian of Eq.(8) into the one of Ref.20.

A plane wave representation can be used for a translationally invariant material in \vec{k} space. It is easy to check that $\Xi H(\vec{k}) \Xi^T = -H(-\vec{k})$. Time reversal transformation provides $\Theta H(\vec{k}) \Theta^T = H(-\vec{k})$ only for $\Delta_p^* =$

$-\Delta_p$. The chirality operator, $\Gamma = i \Xi \Theta$ is such that $\Gamma H(k) \Gamma^{-1} = H(k)$, provided $\Delta_p^* = -\Delta_p$.

Bulk excitation energies, λ , of the Hamiltonian of Eq.(8) are given by ($k^2 = k_{\parallel}^2 + k_z^2$):

$$\lambda^2 = k^2 + M^2 + \mu^2 + |\Delta_p|^2 \pm 2\sqrt{(k_z^2 + M^2)(\mu^2 + |\Delta_p|^2) + k_{\parallel}^2 \mu^2}. \quad (10)$$

Here we have kept only linear terms in the Hamiltonian by dropping the laplacian appearing inside \mathcal{M} (i.e. $\mathcal{M} \rightarrow M$). When the chemical potential is in the conduction band ($\mu > M \gg |\Delta_p|$), we can neglect $|\Delta_p|^2$ in the square root. Defining single particle energies $\eta_{\vec{k}} = \sqrt{M^2 + k^2}$ the eigenenergies take the form:

$$\lambda = \pm \sqrt{(\pm \eta_{\vec{k}} - \mu)^2 + |\Delta_p|^2}. \quad (11)$$

While two of the resulting bands are very far from μ , the other two describe the opening of the superconducting gap at $\eta_{\vec{k}} \sim \mu$.

It is known that at an interface between a TI and a trivial insulator, there are delocalized boundary states. Their energy dispersion is described by a Dirac cone $\epsilon = \pm k_{\parallel}$, where k_{\parallel} lies on the boundary. An interface between a TI and a trivial insulator with the same bulk gap, but with no band inversion, can be easily mimicked just by keeping the linear terms in the derivatives of the model Hamiltonian of Eq.(4) and by changing the sign of M between the two half spaces²⁶. The resulting wavefunctions of the boundary states are concentrated on the plane at $z = 0$ and decay exponentially in the z direction normal to the surface, with the decay length $|M|/\hbar v$. In view of the fact that the linearized model Hamiltonian just reproduces the long wavelength behavior, this boundary condition is good enough¹¹. Analytically continuing the bulk eigenenergies of Eq.(10) with $k_z^2 \rightarrow -\kappa^2 = -M^2$, we get:

$$\lambda = \pm \sqrt{(\pm k_{\parallel} - \mu)^2 + |\Delta_p|^2}, \quad (12)$$

which would describe surface superconductivity with a gap opening at $|\mu| < M$. Again, two of the bands of the excitations are very far off the Fermi energy, while the other two feature the opening of the superconducting gap in the Dirac dispersion of the boundary states.

Even-parity pairing

In the chosen basis of Eq.(7), the uniform order parameter Δ_s , assumed to be independent of the orbital,

as explained in Section II.A, appears as an off diagonal contribution in an Hamiltonian of the following structure:

$$H^s = \begin{pmatrix} H_+^s & S^3 i \partial_z + (C^+ \Delta_s + h.c.) \\ S^3 i \partial_z + (C^+ \Delta_s + h.c.) & H_-^s \end{pmatrix}, \quad (13)$$

where

$$H_{\pm}^s = \begin{pmatrix} \mp \mathcal{M} - \mu & i(\partial_x - i \partial_y) & 0 & 0 \\ i(\partial_x + i \partial_y) & \pm \mathcal{M} - \mu & 0 & 0 \\ 0 & 0 & \pm \mathcal{M} + \mu & i(\partial_x - i \partial_y) \\ 0 & 0 & i(\partial_x + i \partial_y) & \mp \mathcal{M} + \mu \end{pmatrix}. \quad (14)$$

The Hamiltonian in the absence of the vortex has the properties $\Theta H(k) \Theta^{-1} = H(-k)$ and $\Gamma H(k) \Gamma^{-1} = H(k)$, only if Δ_s is real. Again, if the wavefunction decays as $\exp(-|M||z|)$ at the boundary, the spectrum is given by Eq.(12), with $\Delta_p \rightarrow \Delta_s$. An unitary transformation which changes the basis into

$$UB \equiv \left[\psi_{g\uparrow}, \psi_{u\downarrow}, \psi_{g\downarrow}^\dagger, -\psi_{u\uparrow}^\dagger, \psi_{u\uparrow}, \psi_{g\downarrow}, \psi_{u\downarrow}^\dagger, -\psi_{g\uparrow}^\dagger \right]^T \quad (15)$$

transforms the Hamiltonian in the convenient form of Eq.s (9) and (8), with $\Delta_p \rightarrow \Delta_s$.

IV. QUASIPARTICLE STATES BOUND AT A VORTEX LINE

As shown in the previous Section, the even-parity and odd-parity induced proximity give rise to the same matrix form of the model Hamiltonian, but in different bases. While the basis B of Eq.(7) is appropriate for the odd-parity superconducting correlations, giving rise to Eq.s(8),(9), even-parity superconducting order is described by the same Hamiltonian matrix when the basis is UB given by Eq.(15). We now search for zero energy excitations corresponding to quasiparticles bound to a vortex piercing an heterostructure $S/TI/S$ in the form of a slab laying in the $x - y$ plane, with boundary flat planes at $z = 0, L$. Let the axis of the vortex be along the \hat{z} axis. It is appropriate to move to cylindrical coordinates with radial coordinate, r , measured from the vortex singularity and azimuthal angle around the vortex axis, θ . Outside the vortex core, the Hamiltonian now reads:

$$H = \begin{pmatrix} H_+ & S^3 C^3 i \partial_z \\ S^3 C^3 i \partial_z & H_- \end{pmatrix}, \quad (16)$$

where

$$H_{\pm}(r > \xi_o) = \begin{pmatrix} \mp \mathcal{M} - \mu & i e^{-i\theta} \left(\partial_r - \frac{i}{r} \partial_{\theta} - \frac{q}{2r} \right) & \pm \Delta e^{-iq\theta} & 0 \\ i e^{i\theta} \left(\partial_r + \frac{i}{r} \partial_{\theta} + \frac{q}{2r} \right) & \pm \mathcal{M} - \mu & 0 & \pm \Delta e^{-iq\theta} \\ \pm \Delta^* e^{iq\theta} & 0 & \pm \mathcal{M} + \mu & -i e^{-i\theta} \left(\partial_r - \frac{i}{r} \partial_{\theta} + \frac{q}{2r} \right) \\ 0 & \pm \Delta^* e^{iq\theta} & -i e^{i\theta} \left(\partial_r + \frac{i}{r} \partial_{\theta} - \frac{q}{2r} \right) & \mp \mathcal{M} + \mu \end{pmatrix}. \quad (17)$$

Here $q = \pm 1$ is the charge of the vortex and $\xi_o \sim \hbar v / \Delta$ is the radius of the vortex core. Δ stands for Δ_s or Δ_p , depending on the actual superconducting order. We have implemented sign changes with respect to Eq.s(8),(9), to take care of the fact that all of the derivatives should act to the right hand side. We have also added the vector potential associated to the vortex, which, far away from the vortex core, takes the form of a pure singular gauge:

$$A_r = 0, \quad A_{\theta}(r) = -\frac{1}{r} \partial_{\theta} \chi; \quad \chi = q\phi \frac{\theta}{2\pi}, \quad (18)$$

($\phi = hc/2e$ is the flux unit). The phase factor $e^{iq\theta}$ breaks the TR invariance, which holds when Δ_s is real (Δ_p is purely imaginary). The procedure to search for zero energy eigenstates of the Hamiltonian of Eq.(16) is sketched in Appendix A for *s-wave* pairing and in Appendix B for *odd-parity* pairing. In the next Subsections we report the results.

A. *s-wave*, singlet proximity

When proximity induces *s-wave*, singlet superconducting correlations, an axial vortex of charge $q = \pm 1$, binds Majorana quasiparticles at the interface with the topologically trivial superconductors. The zero energy eigenstates are found along the lines sketched in Appendix A, by matching solutions inside and outside the vortex core. Its boundary is defined as a circle of radius $\xi_o \sim \hbar v / \Delta$. An analytic derivation of the quantum field in the two-orbital model can be given far away from the vortex core in the limit of $\mu = 0$ (midgap MBS). Two zero energy real fermion fields, localized far apart at the two boundary surfaces of the slab, $z \sim 0^+$, L^- , take the form, outside the vortex core:

$$\begin{aligned} \gamma(z \sim 0^+) &\propto e^{-\lambda z} K_{1/2}(\Delta_s r) \left\{ \left[e^{-i\theta/2} e^{iq\theta} \psi_{g\uparrow} + e^{i\theta/2} e^{-iq\theta} \psi_{g\uparrow}^{\dagger} \right] + i \left[e^{-i\theta/2} e^{iq\theta} \psi_{u\uparrow} - e^{i\theta/2} e^{-iq\theta} \psi_{u\uparrow}^{\dagger} \right] \right\}, \\ \gamma(z \sim L^-) &\propto e^{-\lambda(L-z)} K_{1/2}(\Delta_s r) \left\{ -i \left[e^{i\theta/2} e^{iq\theta} \psi_{g\downarrow} - e^{-i\theta/2} e^{-iq\theta} \psi_{g\downarrow}^{\dagger} \right] + \left[e^{i\theta/2} e^{iq\theta} \psi_{u\downarrow} + e^{-i\theta/2} e^{-iq\theta} \psi_{u\downarrow}^{\dagger} \right] \right\}, \end{aligned} \quad (19)$$

with $\lambda \sim |M|$ and $z > 0$ in the TI. Here K_{\pm} are the modified Bessel functions: $K_{\pm 1/2}(wr) = e^{-wr} / \sqrt{wr}$, so that the excitations are localized also in the surface plane and the wavefunction is, of course, normalizable. The decay length scale is $w^{-1} \sim \xi_o$.

The two Majorana excitations mix both u and g orbitals and are not eigenstates of the spin. It is shown in Appendix A, Eq.s(A6), that $\gamma(z \sim 0)$ and $\gamma(z \sim L)$ form a time reversed pair, notwithstanding the fact that the vortex breaks TR. This is a confirmation of the fact that they are neutral excitations, not influenced by the presence of the magnetic field. Having a $e^{\pm i\theta/2}$ factor, they change sign when moved along a loop about the vortex singularity, as expected.

Inside the vortex core, the solution requires an r -dependent order parameter $\Delta_s(r)$ together with the corresponding vector potential $A(r)$ and should be matched with the one given previously at $r = \xi_o$. Solutions for any μ within the bulk gap, require complex decay lengths $\lambda = \lambda_1 + i\lambda_2$ in the z -direction. One can impose zero

wavefunction at $z = 0$ and use the fact that, being the wavefunction associated with $\psi_{u\sigma}$ odd in z , a prefactor like the $e^{-\lambda_1 z} \cos \lambda_2 z$ can appear, while, in the case of $\psi_{g\uparrow}$, which corresponds to an even z -function, the prefactor should be of the type $e^{-\lambda_1 z} \sin \lambda_2 z$.

To derive MBSs at finite $|\mu| < M$, all the vector components should be involved and the solution can be found only numerically. At finite μ , states extended along the vortex core can also be conceived. They are also extended on the boundary surface, in the form of circular waves centered at the vortex singularity. In fact, in case the solution is of the form $e^{\pm i\mu z}$, it is easy to see that the Bessel functions solving these equations have to be of the first kind, i.e. $J_{\pm 1/2}(\Delta_s r)$, or $H_{\pm 1/2}^{(1,2)}(\Delta_s r)$, which describe normalizable circular states, delocalized in r . This is briefly shown at the end of Appendix A. These states, which are not neutral fermionic states, could be scattering states of the kind derived in Ref.16, to study coherent transport. Analogous states occur when proximity is *odd-parity* and they will be presented at length

in Appendix B for that case. The appearance of these states may be responsible for the "vortex phase transition", which is expected to destroy the MBS localized at each of the boundaries in $z = 0, L$ ¹⁷.

B. Proximity induced odd-parity pairing

An approach similar to the one of Subsection A can be used in the case of the Hamiltonian of Eq.(16), with

$$\begin{aligned} \Psi_L(r > \tilde{\xi}_o, \theta, z) &\propto e^{-i\kappa z} H_{\frac{1}{2}}^{(2)}(iwr) \cdot \left\{ \left[e^{i\pi/4} e^{i(1+q)\theta/2} \psi_{u\downarrow} + e^{-i\pi/4} e^{-i(1+q)\theta/2} \psi_{u\downarrow}^\dagger \right] \right. \\ &\quad \left. + i \left[e^{-i\pi/4} e^{-i(1-q)\theta/2} \psi_{u\uparrow} + e^{i\pi/4} e^{i(1-q)\theta/2} \psi_{u\uparrow}^\dagger \right] \right\} \\ \Psi_L(r < \tilde{\xi}_o, \theta, z) &\sim e^{-\kappa' z} \xi(\kappa' r) \left\{ \left[e^{i(1+q)\theta/2} \psi_{u\downarrow} - e^{-i(1+q)\theta/2} \psi_{u\downarrow}^\dagger \right] + \left[e^{-i(1-q)\theta/2} \psi_{u\uparrow} + e^{i(1-q)\theta/2} \psi_{u\uparrow}^\dagger \right] \right\}. \end{aligned} \quad (20)$$

Here $z > 0$ in the TI ($M, C, c > 0$) and we have defined $\Delta_p = i\Delta'$ with Δ' real. We get:

$$\begin{aligned} -i\kappa &= -a_1 + i a_2, \quad iw = a_1 - i c a_2 / C \\ a_1 &= \sqrt{(C - c)(M + C\Delta'^2) - C^2\Delta'^2} / (C - c), \\ a_2 &= \frac{C\Delta'}{C - c}, \quad \kappa' = \sqrt{\frac{M}{(C - c)}}, \quad M, C, c > 0, \quad C - c > 0. \end{aligned}$$

This excitation only involves fields referring to u orbitals. By choosing the complementary set of non vanishing components for the vector solution, a SABS arises, which involves the $\psi_{g\sigma}$ fields only. For $r > \tilde{\xi}_o$, the function decaying in z has also an oscillatory component. The function of r is a Hankel function $H_{1/2}^{(2)}(iwr)$ of complex argument and has an oscillator factor $\exp -i a_1$, as well as a decaying exponential factor $\exp -c a_2 / C$. For $r < \tilde{\xi}_o$, $\xi(\kappa' r) \sim H_{1/2}^{(1)}(\kappa' r) + H_{1/2}^{(2)}(\kappa' r)$ is the combination of Hankel functions that converges to zero at the origin (i.e. the point where the order parameter vanishes). In our "hard core" approximation, the value of $\tilde{\xi}_o$ is fixed by matching the two solutions of Eq.(20) at the core boundary.

These behaviors qualify the result as a SABS, which, by inspection, is not a MBS, but a Dirac fermion. It is not an eigenstate of TR and it decays along the vortex core, in an oscillatory fashion. By using the projector $P_L = (1 - \Gamma)/2$ onto the left (L) chiral state ($\Gamma = i\Xi\Theta$ defined in Section III), it is easy to check that the combinations given here in the asymptotic region out of the vortex core, are L -chiral at $\theta = 0$, i.e. of the form: $[\psi_{u\downarrow} + i \psi_{u\uparrow}^\dagger]$ and $[\psi_{u\uparrow} - i \psi_{u\downarrow}^\dagger]$. The partner state to the one given in Eq.(20), which involves the $\psi_{g\sigma}$ field operators, is the R -chiral mate combination at $\theta = 0$, i.e. : $[\psi_{g\uparrow} + i \psi_{g\downarrow}^\dagger], [\psi_{g\downarrow} - i \psi_{g\uparrow}^\dagger]$. The spin expectation value for these quasiparticle states vanishes. However, it is interesting to

odd - parity pairing and zero spin projection along the spin quantization axis which is pinned at the normal to the surface of the slab. In this case, the basis is given by Eq.(7). In a film geometry a midgap solution of the kind of the one given in Eq.(19), localized at the interface and bound to the vortex, can be found. However this is not a MBS. In Appendix B we report the derivation of this zero energy eigenstate in full analogy with the $s - wave$ proximity case. One of these SABS at $\mu = 0$ is:

note that just one spin component has a non vanishing angular momentum around the vortex line. This is, in Eq.(20), the down spin for $q = 1$ or the up spin for $q = -1$, respectively. A similar feature is found in half vortex excitations of the $^3\text{He A}$ -phase²⁷.

Away from the midgap, levels can be found for Dirac Fermion excitations, which correspond to circular waves propagating at the interface inward or outward the vortex singularity and merging into the film by travelling across the slab, along the vortex line, with a radial localization length $\hbar v / \Delta'$. These states involve both u and g orbitals. The $U^{-1}(m = 0) \Psi(r, z)$ field, of spacial dependence $\sim e^{i\kappa z} K_{\pm \frac{1}{2}}(\Delta' r)$, for $r > \tilde{\xi}_o$ in the topologically non trivial slab, is derived in Appendix B [Eq.(B16)]. The inverse length scale $\kappa \approx \sqrt{\mu/c}$ can be found in Eq.(B15). The location of these quasiparticle levels is derived by matching the solution inside the core to the one outside it^{24,28}. The matching fixes the value of μ at which these excitations imply no energy cost. Similar levels split off the conduction or the valence band and reside in the bulk gap. Their orbital structure depends on the symmetry of the bulk band from where the quasiparticle level splits off.

We have checked other choices for the non vanishing vectorial components with no success and we conclude that there is no possibility for a MBS to exist, when proximity induced pairing is *odd - parity*.

V. SUMMARY AND DISCUSSION

Topological insulators hold the promise for future developments in low power spintronics and in quantum computing. It has been argued that Cu doped Bi_2Se_3 could become an odd-parity topological superconductor when μ moves within the conduction band, because of

doping¹⁹. In contrast, doping seems to be unable to induce superconductivity in Bi_2Te_3 ²⁹. Even in the case when the superconductors are conventional metals with s -wave pairing, it can be questioned whether the pairing in the TI is *even-parity* (singlet) or *odd-parity* (triplet) in nature. Therefore, it is interesting to characterize the nature of superconducting proximity at the interface S/TI ³⁰. We have considered a TI slab within the two-orbital model which has been successfully introduced to describe band inversion in various TI, particularly Bi_2Se_3 and Bi_2Te_3 . The two bands are made of orbitals which are even ($\psi_{g\sigma}$) and odd ($\psi_{u\sigma}$) with respect to the surface plane with the normal oriented along the z -axis, which terminates a quintuple layer at $z = 0$. The model Hamiltonian used here is applicable both to a full topological superconducting state and to a TI with superconducting pairing correlations induced by proximity. We perform the analytical matching of the state at the interface between the TI and a superconductor of trivial topology essentially by accounting only for the first order z -derivative appearing in the linearized Hamiltonian. Hence the matching is performed just by imposing the continuity of the wave-function at the interface. In the case of an exponentially decaying wave-function on the topologically non trivial and topologically trivial sides of the interface, this is the same as changing the sign of the mass M at the boundary. This simplified method was shown to provide results²⁶ which are not inconsistent with a full treatment^{11,21}. However, matching conditions at the interface $z = 0$ are a minor concern in this context, as we have dealt with both situations, s -wave proximity and *odd-parity* superconducting pairing, on the same foot. In any other respect, our calculation includes second order derivatives, because they influence drastically the localization of the states in the planar dimensions parallel to the slab. The orbital shape of the *even-parity* order parameter is a combination of $\langle\psi_{g\uparrow}\psi_{g\downarrow}\rangle$ and of $\langle\psi_{u\uparrow}\psi_{u\downarrow}\rangle$, while the *odd-parity* one, similar to a "polar" p-wave, involves the fields $\psi_{g\sigma}, \psi_{u\sigma'}$ multiplied together and zero z -component of the total spin of the pair in the odd-parity case ("polar phase") and we expect that the spin-orbit coupling pins the z -spin quantization axis parallel to the normal to the slab surface. This feature could favor the proximity at the interface with a singlet superconductor and makes the comparison between the *even-parity* singlet and the *odd-parity* superconducting pairing more intriguing, in particular in the search for MBSs. In both cases, superconductivity opens up a gap at the chemical potential, which, in our case is immersed in the Dirac cone dispersion of the boundary states. We have considered the case of an axial vortex piercing the slab along the normal to the surface plane and we have derived analytically the nature of the excitations within the superconducting gap, with a direct comparison of the two superconducting orderings. Accounting for the coordinate z in the quasiparticle wavefunctions, has allowed us to find out whether the bound state is localized at one of the surface boundaries, or it is an extended state along

the vortex axis, across the slab thickness.

Our main concern is to give an answer to the question: when the chemical potential is located in the bulk gap, can MBS appear in the two-orbital TI model, localized at the vortex singularity and squeezed at the interfaces with topologically trivial superconductors, for any type of proximity ordering? The answer is negative, as we briefly sum up, here below.

In the case of the s -wave superconductive proximity, MBSs exist, localized at the vortex singularity. This fact is known since the work by Fu and Kane¹³. It is also known that MBS could survive if the chemical potential penetrates the bulk bands ($|\mu| > M$)¹⁷. Here we recover the same result for $|\mu| < M$, by adopting the two-orbital model. There is one neutral (Majorana) fermion bound to the vortex, localized at each interface (top/bottom) with the superconductors of trivial topology. They decay exponentially away from the interface and do not hybridize, if the slab is thick enough. We give the full expression of the fields for the simplest case that can be handled analytically, i.e. $\mu = 0$, Eq.(19). They mix both orbitals $\psi_{g\sigma}, \psi_{u\sigma'}$ but they do not mix spin projections. It is interesting that the two partner MBS, localized at opposite interfaces, form a time reversed pair, notwithstanding the fact that TR symmetry is broken by the vortex flux. This fact points to the neutral nature of these states.

Away from $\mu = 0$, all the components of the vector spinor are involved and a simple analytical solution cannot be exhibited. However, by examining our analytical derivation it is easy to argue that, if we move the chemical potential to non zero values, the states turn out to be extended Dirac fermions along the vortex axis or propagating waves ingoing or outgoing the vortex singularity. Similar solutions have been studied in Ref.16 as the limiting case of two MBS at opposite interfaces hybridizing significantly, till they eventually delocalize.

In the case of *odd-parity* superconductive proximity, the vortex is unable to bind neutral excitations at the interfaces. To prove our conclusion, we put again the chemical potential at the midgap, $\mu = 0$, and we follow similar steps as the ones that led us to confirm the presence of MBSs in the *even-parity* case. States localized at the two interfaces are SABS, originating from Dirac fermions of opposite chirality. Being SABS excitations, they are localized close to the boundary surfaces and decay inside the slab in an oscillatory way, in the region outside the vortex core (see Eq.(20)). They are also localized close to the vortex core, again in an oscillatory fashion. Within the vortex core, as well as in the topologically trivial superconductor, they have a simple exponential decay. The inverse localization length is $\propto |M|$, while the inverse wavelength of the oscillations is ruled by Δ_p . The expectation value of their spin is again zero, but, in this case, vector components appear of different spin labels, while g and u orbitals are not mixed. This is the reverse of what happens in the *even-parity* case, when spin labels are separated, but different orbitals are

mixed. It is interesting to note that just one spin species in the vector has a non vanishing angular momentum around the vortex line, depending on the charge of the vortex. Being the pair a triplet spin pair, the quasiparticle excitations can acquire features of the ^3He quantum liquid. In Appendix B, we also derive scattering solutions away from midgap, in the form of circular waves localized at the interfaces and propagating from one interface to the other, by ingoing or outgoing the vortex singularity. Their orbital structure depends on the symmetry of the bulk band from where the quasiparticle level splits off.

Having checked other possibilities, we conclude that there is no chance of having protected Majorana excitations with *odd - parity* superconductive proximity and zero projection of the pair spin, in the two-orbital model.

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Appendix A: Zero energy modes for a TI slab with s - wave superconductivity induced by proximity

The matrices H_{\pm}^s of Eq.(17) can be rotated by using the projector on angular momentum eigenvectors³¹ $U_m(q, \theta) = e^{-im\theta} \times e^{-iS^3\theta/2} e^{iqC^3\theta/2}$, to get rid of the explicit θ dependence. Modes appear in pair of states of angular momentum $\pm m$ except for $m = 0$. Therefore we must restrict our search for Majorana modes to the subspace at $m = 0$. We also get $U_0^{-1}(q, \theta)_{jj} \nabla_{jj}^2 U_0(q, \theta)_{jj} \equiv (\partial_r + \frac{1}{2r})^2 + \partial_z^2$, independent of j . Thus (we put $\hbar v = 1$ for the time being),

$$U_0^{-1}(q, \theta) H_{\pm} U_0(q, \theta) = \begin{pmatrix} \mp \mathcal{M} - \mu & i \partial_p & \Delta & 0 \\ i \partial_m & \pm \mathcal{M} - \mu & 0 & \Delta \\ \Delta^* & 0 & \pm \mathcal{M} + \mu & -i \partial_m \\ 0 & \Delta^* & -i \partial_p & \mp \mathcal{M} + \mu \end{pmatrix}. \quad (\text{A1})$$

with derivatives ∂_p, ∂_m including the vector potential, acting on the right hand side. Outside the core of the vortex, in the asymptotic region, they coincide, as they become:

$$\partial_p \rightarrow \partial_r + \frac{(1+q)}{2r} - \frac{q}{2r}; \quad \partial_m \rightarrow \partial_r + \frac{(1-q)}{2r} + \frac{q}{2r}. \quad (\text{A2})$$

It appears clearly that, well away from the vortex core, where the choice of A_{θ} given in Eq.(18) holds, the vector potential drops out of the Hamiltonian.

Let us search for a zero energy eigenvector of the Hamiltonian of Eq.(16) of the form $f(z) \cdot [\xi_1, \eta_2, \xi'_2, \eta'_1] - \eta_1, \xi_2, -\eta'_2, \xi'_1]^T$. where $f(z)$ decays on both sides away from $z = 0$ and ξ, η 's are complex functions of r . If $\xi_2, \xi'_2, \eta_2, \eta'_2$ are taken to be zero, the eight equations that we get reduce to:

$$\begin{aligned} 1 : & -(\mathcal{M} + \mu) \xi_1 - i \partial_z \eta_1 = 0, \\ 2 : & i \partial_m \xi_1 + \Delta_s \eta'_1 = 0, \\ 3 : & -i \partial_m \eta'_1 + \Delta_s^* \xi_1 = 0, \\ 4 : & -(\mathcal{M} - \mu) \eta'_1 + i \partial_z \xi'_1 = 0, \\ 5 : & -(\mathcal{M} - \mu) \eta_1 + i \partial_z \xi_1 = 0, \\ 6 : & -i \partial_m \eta_1 + \Delta_s \xi'_1 = 0, \\ 7 : & -i \partial_m \xi'_1 - \Delta_s^* \eta_1 = 0, \\ 8 : & (\mathcal{M} + \mu) \xi'_1 + i \partial_z \eta'_1 = 0. \end{aligned} \quad (\text{A3})$$

If Δ_s is real, a solution can be obtained with $f(z) = e^{-\lambda z}$ ($z > 0$ and $\lambda > 0$). We only report the $\mu = 0$ case which is analytically straightforward: $\xi'_1 = -\xi_1 = -K_{-1/2}(\Delta_s r)$ and $\eta_1 = -\eta'_1 = -i K_{1/2}(\Delta_s r)$, where K_{\pm} are the modified Bessel functions: $K_{\pm 1/2}(wr) = e^{-wr}/\sqrt{wr}$. The latter functions, together with the exponentially decaying $f(z)$, are also eigenstates of the operator

$$\mathcal{M} = \left\{ M + C \left(\partial_r^2 + \frac{1}{2r} \partial_r - \frac{1}{4r^2} \right) + c \partial_z^2 \right\} \quad (\text{A4})$$

with eigenvalue $\lambda \sim M$, that should be thus consistently determined. According to the given basis of Eq.(15), the linear combination of fields is a real fermion ($z > 0$):

$$\propto e^{-\lambda z} K_{\pm 1/2}(\Delta_s r) \cdot \{ [\psi_{g\uparrow} + \psi_{g\uparrow}^\dagger] + i [\psi_{u\uparrow} - \psi_{u\uparrow}^\dagger] \}$$

The opposite choice: $\xi_1 = \xi'_1 = \eta_1 = \eta'_1 = 0$, provides one solution decaying at the opposite side of the slab $z \sim L$, in the form: $e^{-\lambda(L-z)}$. Undoing the U rotation and gauging away the vector potential, we obtain two real fermion fields. They mix both orbitals and have vanishing expectation value of the spin projection. The spinor part is of the form, outside the vortex core:

$$\begin{aligned} \gamma(z \sim 0) & \rightarrow \left[-e^{-i\theta/2} e^{i\pi/4} e^{iq\theta} \psi_{g\uparrow}, 0, 0, e^{i\theta/2} e^{-i\pi/4} e^{-iq\theta} \psi_{u\uparrow}^\dagger \middle| e^{-i\theta/2} e^{i\pi/4} e^{iq\theta} \psi_{u\uparrow}, 0, 0, e^{i\theta/2} e^{-i\pi/4} e^{-iq\theta} \psi_{g\uparrow}^\dagger \right]^T \\ \gamma(z \sim L) & \rightarrow \left[0, -e^{i\theta/2} e^{-i\pi/4} e^{-iq\theta} \psi_{u\downarrow}, e^{-i\theta/2} e^{i\pi/4} e^{iq\theta} \psi_{g\downarrow}^\dagger, 0 \middle| 0, e^{i\theta/2} e^{-i\pi/4} e^{-iq\theta} \psi_{g\downarrow}, e^{-i\theta/2} e^{i\pi/4} e^{iq\theta} \psi_{u\downarrow}^\dagger, 0 \right]^T. \end{aligned} \quad (\text{A5})$$

We have explicitly included the field operator, from the basis of Eq.(15), and we have put a slash in the middle of the spinor to highlight the internal structure of the states: $\gamma(z \sim 0) \rightarrow [\alpha|\beta]$ and $\gamma(z \sim L) \rightarrow [\alpha'|\beta']$. It can be seen that the α' component and the β one are related by TR, as well as the α component and the β' one. This is to underline that the two Majorana states form a TR pair, notwithstanding the fact that the vortex breaks TR symmetry. By adding the field components together, we get the final expressions or the MBSs reported in Eq.s (19).

At finite μ a solution of Eq.s (A3) is also possible, propagating along the vortex axis as $f(z) \sim e^{\pm i\mu z}$. Again we require $\mathcal{M}[\xi, \eta] = 0$ to satisfy Eq.s (A3) 1,4,5,8. However, at difference with the $\mu = 0$ case worked out above, in which $\xi'_1 = -\xi_1$ were real and $\eta_1 = -\eta'_1$ were purely imaginary, all the vector components will now be real. It follows that the Bessel functions solving Eq.s (A3) 2,3,6,7 have an argument which is purely imaginary. This shows

that they turn into extended waves $\sim J_{\pm 1/2}(\Delta_s r)$ or $\sim H_{\pm 1/2}^{(1,2)}(\Delta_s r)$, as mentioned at the end of Section IV.A.

Appendix B: Zero energy modes for odd – parity superconductive proximity

a) SABSs at $\mu = 0$

Here we tackle first the midgap, zero energy eigenstate in close correspondence with the $\mu = 0$ MBS of the s – $wave$ pairing, presented in Appendix A. We will show that the solution is a Dirac fermion, that can be qualified as a SABS. Following the same procedure as for the s – $wave$ proximity, we start from the Hamiltonian of Eq.(16) in cylindrical coordinates. A generic form of the solution is:

$$f(z) \times \left[\eta_1 e^{-i\chi}, \xi_1 e^{-i\chi'}, -\xi'_1 e^{i\chi'}, \eta'_1 e^{i\chi} | -\eta_2 e^{i\chi}, \xi_2 e^{i\chi'}, \xi'_2 e^{-i\chi'}, \eta'_2 e^{-i\chi} \right]^T. \quad (B1)$$

The equations solved by the zero energy mode are:

$$\begin{aligned} 1: & -(M + \mu) \eta_1 e^{-i\chi} + i\partial_p \xi_1 e^{-i\chi'} - \Delta \xi'_1 e^{i\chi'} - i\partial_z \eta_2 e^{i\chi} = 0, \\ 2: & i\partial_m \eta_1 e^{-i\chi} + (M - \mu) \xi_1 e^{-i\chi'} + \Delta \eta'_1 e^{i\chi} - i\partial_z \xi_2 e^{i\chi'} = 0, \\ 3: & \Delta^* \eta_1 e^{-i\chi} - (M + \mu) \xi'_1 e^{i\chi'} - i\partial_m \eta'_1 e^{i\chi} - i\partial_z \xi'_2 e^{-i\chi'} = 0, \\ 4: & \Delta^* \xi_1 e^{-i\chi'} + i\partial_p \xi'_1 e^{i\chi'} - (M - \mu) \eta'_1 e^{i\chi} + i\partial_z \eta'_2 e^{-i\chi} = 0, \\ 5: & i\partial_z \eta_1 e^{-i\chi} - (M - \mu) \eta_2 e^{i\chi} + i\partial_p \xi_2 e^{i\chi'} - \Delta \xi'_2 e^{-i\chi'} = 0, \\ 6: & -i\partial_z \xi_1 e^{-i\chi'} - i\partial_m \eta_2 e^{i\chi} - (M + \mu) \xi_2 e^{i\chi'} - \Delta \eta'_2 e^{-i\chi} = 0, \\ 7: & i\partial_z \xi'_1 e^{i\chi'} + \Delta^* \eta_2 e^{i\chi} - (M - \mu) \xi'_2 e^{-i\chi'} - i\partial_m \eta'_2 e^{-i\chi} = 0, \\ 8: & i\partial_z \eta'_1 e^{i\chi} - \Delta^* \xi_2 e^{i\chi'} - i\partial_p \xi'_2 e^{-i\chi'} + (M + \mu) \eta'_2 e^{-i\chi} = 0. \end{aligned} \quad (B2)$$

We search for a real eigenvector. As $\Delta_p \sim \langle \psi_{g\uparrow} \psi_{u\downarrow} \rangle$ the choice $\chi = \chi'$ is consistent with the *odd – parity* pairing, as it gives a purely imaginary $\Delta_p \equiv i \Delta'$ (which defines Δ' as real). By posing $f(z) = e^{-i\kappa z}$ (with complex κ), the appropriate "would be MBS " requires that, e.g., $\eta_1 = \eta'_1 = \xi_2 = \xi'_2 = 0$. In this case, from Eq.s(B2) we get:

$$\begin{aligned} 2: & (\mathcal{M} - \mu) \xi_1 = 0 & 5: & -(\mathcal{M} - \mu) \eta_2 = 0 \\ 3: & -(\mathcal{M} + \mu) \xi'_1 = 0 & 8: & (\mathcal{M} + \mu) \eta'_2 = 0, \end{aligned} \quad (B3)$$

which implies that, at $\mu = 0$, the surviving vector components should be eigenstates of the operator \mathcal{M} with

zero eigenvalue. Besides:

$$\begin{aligned} 1: & i\partial_p \xi_1 - i \Delta' \xi'_1 - \kappa \eta_2 = 0 \\ 4: & i \partial_p \xi'_1 - i \Delta' \xi_1 + \kappa \eta'_2 = 0 \\ 6: & -i \partial_m \eta_2 - i \Delta' \eta'_2 - \kappa \xi_1 = 0 \\ 7: & -i \partial_m \eta'_2 - i \Delta' \eta_2 + \kappa \xi'_1 = 0 \end{aligned} \quad (B4)$$

A possible choice is $\xi_1 = i \xi$, $\eta_2 = i \eta$ and $\xi'_1 = \eta$, $\eta'_2 = \xi$. This implies that the equations become:

$$\begin{aligned} 1: & \partial_p \xi + i (\kappa + \Delta') \eta = 0, \\ 4: & \partial_p \eta - i (\kappa + \Delta') \xi = 0, \\ 6: & \partial_m \eta - i (\kappa + \Delta') \xi = 0 \\ 7: & \partial_m \xi + i (\kappa + \Delta') \eta = 0. \end{aligned} \quad (B5)$$

Rewriting $\partial = \partial_r + 1/2r$ in place of $\partial_{m,p}$, because, asymptotically out of the vortex core, the two operators coincide, a good solution out of the vortex core,

satisfying convergency for large r is: $\eta \sim H_{-1/2}^{(1)}(-iwr)$ and $\xi \sim H_{1/2}^{(1)}(-iwr)$ with $w = (\kappa + \Delta')$. The latter turns out to be a complex parameter. Observing that $(\partial - \alpha^2) H_{\pm 1/2}^{(1)}(\alpha r) = 0$, the constraint $\mathcal{M} H_{\pm 1/2}^{(1)} = 0$ is fulfilled, provided $M + C w^2 - c \kappa^2 = 0$, what gives $(a_1 = (a + a^*)/2, a_2 = (a - a^*)/2)$:

$$\begin{aligned} -i\kappa &= -a_1 + i a_2, \quad -iw = -a_1 + i c a_2 / C \\ a_1 &= \sqrt{(C - c)(M + C \Delta'^2) - C^2 \Delta'^2} / (C - c), \\ a_2 &= \frac{C \Delta'}{C - c}, \quad M, C, c > 0, \quad C - c > 0. \end{aligned}$$

Accordingly, $f(z)$ is a decaying function of z , but it also has an oscillatory component. The function of r can be put in the form of the Hankel function $H_{1/2}^{(2)}(iwr)$ and has an oscillator factor $\exp -i a_1$, as well as a decaying exponential $\exp -c a_2 / C$. This behavior qualifies the state to be a SABS, which is not a MBS, however. In fact, being the wavefunction of the form $[0, i \xi, \xi, 0 | i \xi, 0, 0, -\xi]$, in the basis of Eq.(7) the field operator of the excitation is:

$$e^{-i\kappa z} H_{\frac{1}{2}}^{(2)}(iwr) \cdot \left\{ \left[e^{i\pi/4} \psi_{u\downarrow} + e^{-i\pi/4} \psi_{u\downarrow}^\dagger \right] + i \left[e^{-i\pi/4} \psi_{u\uparrow} + e^{i\pi/4} \psi_{u\uparrow}^\dagger \right] \right\}. \quad (\text{B6})$$

which correspond to a mid gap SABS.

Inside the vortex core we do not require an exponential decay in r , but a zero of the wavefunction at $r = 0$. Let us define $f(z) = e^{-\kappa' z}$ for $z > 0$. In this case the Eqs(B2) become:

$$\begin{aligned} 1: \quad & [i \partial_p \xi_1 - i \Delta' \xi'_1 + i \kappa' \eta_2] = 0 \\ 4: \quad & [i \partial_p \xi'_1 - i \Delta' \xi_1 - i \kappa' \eta'_2] = 0 \\ 6: \quad & [-i \partial_m \eta_2 - i \Delta' \eta'_2 + i \kappa' \xi_1] = 0 \\ 7: \quad & [-i \partial_m \eta'_2 - i \Delta' \eta_2 - i \kappa' \xi'_1] = 0. \end{aligned} \quad (\text{B7})$$

Here $\partial_{p,m}$ do not have the form of Eq.(A2), because, the vector potential does not take the asymptotic expression typical of a singular gauge within the vortex core. Also $\Delta'(r)$ is expected to have a linear r -dependence close to the vortex axis²⁴. A simple way to approximate the eigenfunction is to consider a hard core with $\Delta' = 0$ and to overlook the vector potential difference. By putting $\eta'_2 = \xi_1 \equiv \xi$ and $\xi'_1 = \eta_2 \equiv \eta$ we get the two equations:

$$\begin{aligned} 1(\sim 7): \quad & \partial \xi + \kappa' \eta = 0 \\ 4(\sim 6): \quad & \partial \eta - \kappa' \xi = 0 \\ \rightarrow \quad & \partial^2 \xi = -\kappa'^2 \xi, \quad \partial^2 \eta = -\kappa'^2 \eta. \end{aligned} \quad (\text{B8})$$

Appropriate solutions of Eqs.(B8) are the Hankel functions $H_{\pm 1/2}^{(1,2)}(\kappa' r)$. Besides

$$\mathcal{M} [\xi, \eta] = 0 \rightarrow M - (C - c) \kappa'^2 = 0, \quad (\text{B9})$$

what defines κ' as a real parameter for $M, C - c > 0$. Having two possible normalizable solutions, we can impose that their combination vanishes at $r = 0$. We thus obtain $\xi(r)$ and $\eta(r)$:

$$\begin{aligned} \xi(r) &= H_{+1/2}^{(1)}(\alpha r) + H_{+1/2}^{(2)}(\alpha r), \\ \eta(r) &= -i \left\{ H_{-1/2}^{(1)}(\alpha r) - H_{-1/2}^{(2)}(\alpha' r) \right\}, \end{aligned} \quad (\text{B10})$$

which are real for any r . They converge to zero at the origin and are such that $\eta(r) = \xi(r)$. In conclusion, inside the core, the vector is:

$$\sim e^{-\kappa' z} [0, \xi(r), -\eta(r), 0 | -\eta(r), 0, 0, \xi(r)]^T. \quad (\text{B11})$$

In the basis of Eq.(7), this gives:

$$\sim \xi(r) e^{-\kappa' z} \left\{ (\psi_{u\downarrow} - \psi_{u\downarrow}^\dagger) - (\psi_{u\uparrow} + \psi_{u\uparrow}^\dagger) \right\}. \quad (\text{B12})$$

Undoing the transformation $U(m=0)$ we get Eq.(20) of the text. Our "hard core" approximation depends on the parameter $\tilde{\xi}_o$, which fixes the core boundary. The value of $\tilde{\xi}_o$ can be determined by matching the inside and the outside solutions at the core boundary.

By choosing $\eta_1, \eta'_1, \xi_2, \xi'_2$ non zero and the other spinorial components zero, the partner state of Eq.(20), involving the $\psi_{g\sigma}$ fields, can be found.

b) extended states along the vortex line

Here we show that a Fermi Dirac state solution is also possible, involving real vector components, which describe a wave travelling along the vortex, with z -dependence $e^{i\kappa z}$ and κ real. This is a bound state within the bulk gap. All the $\xi's, \eta's$ have to be non vanishing. If we pose $\eta'_1 = -\eta_2$, $\eta'_2 = -\eta_1$ and $\xi'_1 = \xi_2$, $\xi'_2 = \xi_1$, the Eq.(B2) can be solved with:

$$\begin{aligned} \xi_1 &= K_{1/2}(\Delta' r), \quad \xi_2 = K_{-1/2}(\Delta' r) \\ \eta_1 &= K_{1/2}(\Delta' r), \quad \eta_2 = K_{-1/2}(\Delta' r) \end{aligned} \quad (\text{B13})$$

by requiring the following :

$$1: -(\mathcal{M} + \mu)\eta_1 + \kappa \eta_2 = 0 \quad 5: -(\mathcal{M} - \mu)\eta_2 - \kappa \eta_1 = 0$$

provided

$$1': \partial_p \xi_1 - \Delta' \xi'_1 = 0 \quad 5': \partial_p \xi_2 - \Delta' \xi'_2 = 0$$

The first two are compatible, if, after substitution of the eigenvalue to the operator \mathcal{M} , the determinant: $\mathcal{M}^2 - \mu^2 + \kappa^2 = 0$ vanishes. Similarly:

$$8: -(\mathcal{M} + \mu)\eta'_2 + \kappa \eta'_1 = 0 \quad 4: -(\mathcal{M} - \mu)\eta'_1 - \kappa \eta'_2 = 0$$

provided

$$8': \partial_p \xi'_2 - \Delta' \xi_2 = 0 \quad 4': \partial_p \xi'_1 - \Delta' \xi_1 = 0$$

$$6: -(\mathcal{M} + \mu)\xi_2 + \kappa \xi_1 = 0 \quad 2: (\mathcal{M} - \mu)\xi_1 + \kappa \xi_2 = 0$$

provided

$$6': -i \partial_m \eta_2 + \Delta' \eta'_2 = 0 \quad 2': \partial_m \eta_1 + \Delta' \eta'_1 = 0$$

$$3: -(\mathcal{M} + \mu)\xi'_1 + \kappa \xi'_2 = 0 \quad 7: (\mathcal{M} - \mu)\xi'_2 + \kappa \xi'_1 = 0$$

provided

$$3': \partial_m \eta'_1 + \Delta' \eta_1 = 0 \quad 7': \partial_m \eta'_2 + \Delta' \eta_2 = 0$$

The value of κ for any μ is fixed by solving the determinantal condition:

$$\left[M + C\Delta'^2 - c\kappa^2\right]^2 + (\hbar v \kappa)^2 - \mu^2 = 0 \quad (\text{B14})$$

($\hbar v$ has been restored). By posing $m = M + C\Delta'^2$ we get:

$$\kappa^2 = \frac{m}{c} - \frac{(\hbar v)^2}{2c^2} + \frac{1}{c} \sqrt{\mu^2 - \frac{m(\hbar v)^2}{c} + \frac{(\hbar v)^4}{4c^2}} \quad (\text{B15})$$

which gives a real κ , because³²: $m > (\hbar v)^2/2c$. The plus sign in front of the square root of Eq.(B15) has been chosen on physical grounds, in order to obtain that the wavelength of the propagation along z increases when μ goes deeper down in the bulk gap. Qualitatively is $\kappa \sim \sqrt{\mu/c}$. The matching between the inside and the outside of the vortex core will fix the value of μ at which the excitation is zero energy²⁸. Hence the location of the energy level is fully determined.

Just outside the slab, at the interface with the topologically trivial material is $m < 0$, so that $\kappa^2 < 0$ and the solution decays with z away from the surface. Being κ purely imaginary, the vector components of the eigenfunction acquires alternatively an i factor, as is the case leading to Eq.(B6). This changes the Bessel functions $K_{\pm 1/2}$ into the corresponding Hankel functions. While in the case of $\mu = 0$ these functions were localized in r anyhow, because of complex argument, in this case they are delocalized in r , because the argument, $\Delta' r$, is real. This implies that, at the interface, the waves become circularly propagating inward or outward the vortex line.

The consequence is that this state is a traveling wave along the vortex axis in the non trivial topological material, while it decays outside at the interface with the trivial material and propagates outward or inward at the surface. Its energy is localized in the gap. This is not a Majorana state, however. Using Eq.(B1) for the vector, we obtain:

$$[\eta_1, \xi_1, -\xi_2, -\eta_2] - [\eta_2, \xi_2, \xi_1, -\eta_1] \rightarrow [\eta, \xi, -\xi, -\eta] - [\eta, \xi, \xi, -\eta]$$

which, in the basis of Eq.(7), provides:

$$\rightarrow U^{-1}(m=0) \Psi(r, z) \propto e^{i\kappa z} K_{\pm \frac{1}{2}}(\Delta' r) \cdot [(\psi_{g\uparrow} + \psi_{g\uparrow}^\dagger) + (\psi_{u\downarrow} - \psi_{u\downarrow}^\dagger) + (\psi_{u\uparrow} + \psi_{u\uparrow}^\dagger) + (\psi_{g\downarrow} + \psi_{g\downarrow}^\dagger)] \quad (\text{B16})$$

The final step would be undoing the transformation $U(m=0)$.

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